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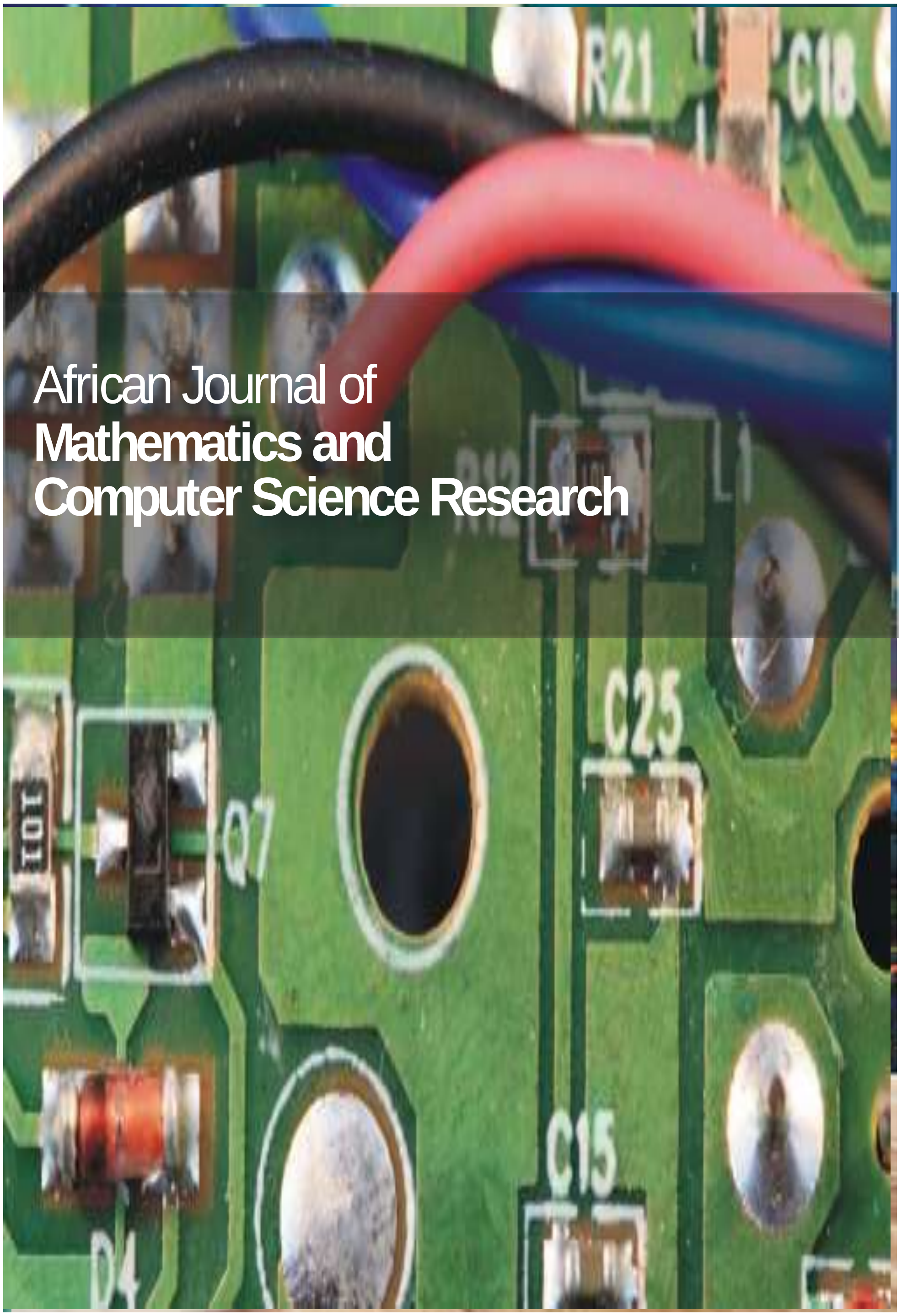

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Full Length Research Paper

Multi parameter fuzzy soft set approach to decision making problem

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In this paper, a new approach for decision making problem is introduced by extending the definition of fuzzy soft set for multiple parameter sets and is called extended fuzzy soft set. Also, few operations such as “AND” and “ $MaxMin$ ” are defined on extended fuzzy soft sets and illustrated with examples. Further an algorithm for decision making using the concept of extended fuzzy soft set is presented. The decision making process includes construction of comparison matrix and ranking strategy is based on the row sum of comparison matrix. Finally an application of proposed algorithm for decision making is presented.

Key words: Fuzzy soft set, decision making, comparison matrix, ranking.

INTRODUCTION

To handle the uncertainties in the imprecise data arising in most of real life problems in engineering, social and medical science, economics, environment, etc., Zadeh (1965) introduced the concept of fuzzy set and fuzzy set operations and further various researchers have extended it to an intuitionistic fuzzy set (Atanassov and Gargov, 1989), interval intuitionistic fuzzy set (Atanassov, 1986) in which information to handle the uncertain information is more precise. Under these environments, different types of approaches are discussed by the researchers to solve the decision-making problems (Xu, 2007; Garg, 2018a, b; Xu and Yager, 2006; Garg, 2017; Garg and Kumar, 2018). As there is an inadequacy of the parameterizations tool associated with these approaches, Molodtsov (1999) introduced soft sets as general mathematical tool for dealing with objects which have been defined using a very general set of characteristics and applied the soft theory into several directions, such as, game theory, operations research, Riemann integration, theory of probability, theory of measurement, and so on. Maji et al. (2002) presented application of soft sets in decision making problems. Maji et al. (2001) introduced the concept of the fuzzy soft sets by using the ideas of fuzzy sets (Zadeh, 1965) and then many interesting applications of fuzzy soft set theory have been proposed by various researchers. Roy and Maji (2007) presented applications of fuzzy soft sets for decision making problem. Som (2006) defined soft relation and fuzzy soft relation on the theory of soft sets. Mukherjee and Chakraborty (2008) worked on intuitionistic fuzzy soft

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relations. Aktas and Cagman ((2007) compared soft sets with the related concepts of fuzzy sets and rough sets. Yang et al. (2007) worked on operations on fuzzy soft sets. Zou and Xiao (2008) introduced the soft set and fuzzy soft set into the incomplete environment. Yang et al. (2009) presented the combination of interval-valued fuzzy set and soft set. Kong et al. (2008) introduced the normal parameter reduction in the fuzzy soft sets. Majumdar and Samanta (2010) presented generalized fuzzy soft sets for decision making problems. Zhao and Jia (2015) presented decision making method based on Cartesian products of fuzzy soft sets. Garg et al. (2016) defined the notion of the fuzzy number intuitionistic fuzzy soft sets. However, for solving the decision making problems, various researchers have utilized different aggregation operators (Garg and Arora, 2018a, b, c; Arora and Garg, 2018a, b) and information measures (Garg and Arora, 2017a, b; Mukherjee and Sarkar, 2014; Rajarajeswari and Dhanalakshmi, 2014; Arora and Garg, 2018c) under the soft set environment. In this paper, a new approach for decision making problem is introduced by extending the definition of fuzzy soft set for multiple parameter sets and is called as extended fuzzy soft set. Also, few operations such as "AND" and "$MaxMin$" are defined on extended fuzzy soft sets and illustrated with examples. Finally an algorithm for decision making was presented and the decision making process includes construction of comparison matrix and ranking strategy based on the row sum of comparison matrix (Roy and Maji, 2007; Kong et al., 2009).

BASIC DEFINITIONS

Definition 1

A fuzzy set A of a non empty set X is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ represents "degree of membership" of x in A, for $x \in X$ and I^X represents family of all fuzzy sets on X (Zadeh, 1965).

Definition 2

Let X be an initial universe and E be a set of parameters, a pair (F, A) denoted by F_A for $A \subseteq E$ is called fuzzy soft set, where F is mapping given by $F : A \to I^X$ (Maji et al., 2001).

Definition 3

The Cartesian "AND" product of two fuzzy soft sets F_A and F_B over a common universe X denoted by $H_C = F_A \wedge F_B$, is defined as $H_C : A \times B \to I^X$ and $H_C(a,b) = F_A(a) \wedge F_B(b)$, where $(a,b) \in A \times B$ (Zhao and Jia, 2015).

Definition 4

Comparison matrix is a square matrix (c_{ij}) in which rows and columns are labeled by the object names of the universe and the entries $c_{ij} = \sum_{k=1}^{m} (\alpha_{ik} - \alpha_{jk})$ where α_{ik} is the membership value of i^{th} object for k^{th} parameter (Roy and Maji, 2007; Kong et al., 2009).

EXTENDED FUZZY SOFT SETS

Here, fuzzy soft set definition is extended for two parameter sets and named it as extended fuzzy soft set. Suppose X is an initial universe and E and K are primary and secondary set of parameters. Let I^X denote family of all fuzzy sets over X and E^X denote family of all fuzzy soft sets over X with respect to the parameter set E. For any $A \subseteq K$, a pair (F^*, A) denoted by F_A^* is called extended fuzzy soft set over X, where F^* is a mapping given by $F^* : A \to E^X$ defined by $F_A^*(k) = F_{E_A}(k)$ defined by $F_{E_A}(k) = \phi$ if $k \notin A$ and $F_{E_A}(k) \neq \phi$ if $k \in A$.

Example 1

Consider a universal set $X = \{x_1, x_2, x_3, x_4\}$, primary parameter set $E = \{e_1, e_2, e_3\}$ and secondary parameter set $K = \{k_1, k_2, k_3, k_4\}$ and let $A = \{k_1, k_3\}$ and $B = \{k_2, k_3\}$. Define extended fuzzy soft sets as $F_A^* = \left\{(k_1, F_{E_A}(k_1)), (k_3, F_{E_A}(k_3))\right\}$ and $F_B^* = \left\{(k_2, F_{E_B}(k_2)), (k_3, F_{E_B}(k_3))\right\}$, where

$$F_{E_A}(k_1) = \left\{e_1 = \left\{\frac{0.1}{x_1}, \frac{0.3}{x_2}, \frac{0.4}{x_3}, \frac{0.7}{x_4}\right\}, e_2 = \left\{\frac{0.2}{x_1}, \frac{0.4}{x_2}, \frac{0.6}{x_3}, \frac{0.4}{x_4}\right\}, e_3 = \left\{\frac{0.8}{x_1}, \frac{0.7}{x_2}, \frac{0.9}{x_3}, \frac{0.1}{x_4}\right\}\right\}$$

$$F_{E_A}(k_3) = \left\{e_1 = \left\{\frac{0.2}{x_1}, \frac{0.6}{x_2}, \frac{0.5}{x_3}, \frac{0.8}{x_4}\right\}, e_2 = \left\{\frac{0.1}{x_1}, \frac{0.7}{x_2}, \frac{0.3}{x_3}, \frac{0.6}{x_4}\right\}, e_3 = \left\{\frac{0.8}{x_1}, \frac{0.6}{x_2}, \frac{0.4}{x_3}, \frac{0.5}{x_4}\right\}\right\}$$

Table 1. Tabular representation of extended fuzzy soft sets.

$F_{E_A}(k_1)$	x_1	x_2	x_3	x_4
e_1	0.1	0.3	0.4	0.7
e_2	0.2	0.4	0.6	0.4
e_3	0.8	0.7	0.9	0.1
$F_{E_A}(k_3)$	x_1	x_2	x_3	x_4
e_1	0.2	0.6	0.5	0.8
e_2	0.1	0.7	0.3	0.6
e_3	0.8	0.6	0.4	0.5
$F_{E_B}(k_2)$	x_1	x_2	x_3	x_4
e_1	0.3	0.5	0.6	0.8
e_2	0.1	0.9	0.4	0.7
e_3	0.1	0.6	0.4	0.6
$F_{E_B}(k_3)$	x_1	x_2	x_3	x_4
e_1	0.2	0.4	0.6	0.7
e_2	0.2	0.5	0.6	0.8
e_3	0.9	0.8	0.4	0.2

$$F_{E_B}(k_2) = \left\{ e_1 = \left\{ \frac{0.3}{x_1}, \frac{0.5}{x_2}, \frac{0.6}{x_3}, \frac{0.8}{x_4} \right\}, e_2 = \left\{ \frac{0.1}{x_1}, \frac{0.9}{x_2}, \frac{0.4}{x_3}, \frac{0.7}{x_4} \right\}, e_3 = \left\{ \frac{0.1}{x_1}, \frac{0.6}{x_2}, \frac{0.4}{x_3}, \frac{0.6}{x_4} \right\} \right\}$$

$$F_{E_B}(k_3) = \left\{ e_1 = \left\{ \frac{0.2}{x_1}, \frac{0.4}{x_2}, \frac{0.6}{x_3}, \frac{0.7}{x_4} \right\}, e_2 = \left\{ \frac{0.2}{x_1}, \frac{0.5}{x_2}, \frac{0.6}{x_3}, \frac{0.8}{x_4} \right\}, e_3 = \left\{ \frac{0.9}{x_1}, \frac{0.8}{x_2}, \frac{0.4}{x_3}, \frac{0.2}{x_4} \right\} \right\}$$

The above extended fuzzy soft sets can also be represented in a tabular form as shown below and throughout the paper following representation has been used for representing extended fuzzy soft sets and also similar representation for fuzzy soft as well as fuzzy sets shown in Table 1.

Definition 5

The Cartesian "AND" product of two extended fuzzy soft sets F_A^* and F_B^* over a common universe X denoted by $H_C^* = F_A^* \wedge F_B^*$, is defined as $H_C^* : A \times B \to E^X$ and $H_C^*(a,b) = F_{E_A}(a) \wedge F_{E_B}(b)$, where $(a,b) \in A \times B$.

Example 2

The Cartesian "AND" product of F_A^* and F_B^* (as defined in Example 1) is given by

$$H_C^* = F_A^* \wedge F_B^* = \left\{ \begin{array}{l} \{(k_1 k_2)(F_{E_A}(k_1) \wedge F_{E_B}(k_2))\}, \{(k_1 k_3)(F_{E_A}(k_1) \wedge F_{E_B}(k_3))\}, \\ \{(k_3 k_2)(F_{E_A}(k_3) \wedge F_{E_B}(k_2))\}, \{(k_3 k_3)(F_{E_A}(k_3) \wedge F_{E_B}(k_3))\} \end{array} \right\} \quad (1)$$

where Table 2 shows extended fuzzy soft set H_C^*.

Definition 6

The $MaxMin$ operators on "AND" products of two extended fuzzy soft sets F_A^* and F_B^* are defined as:

$$Max_a Min_b [H_C^*(a,b)] = \vee_{a \in A} \{ \wedge_{b \in B} (F_A^*(a) \wedge F_B^*(b)) \} \quad (2)$$

$$Max_b Min_a [H_C^*(a,b)] = \vee_{b \in B} \{ \wedge_{a \in A} (F_A^*(a) \wedge F_B^*(b)) \} \quad (3)$$

Example 3

The $MaxMin$ operations on "AND" product

Table 2. Extended fuzzy soft set H_C^*.

$(k_1\ k_2)$	x_1	x_2	x_3	x_4	$(k_1\ k_3)$	x_1	x_2	x_3	x_4
$(e_1\ e_1)$	0.1	0.3	0.4	0.7	$(e_1\ e_1)$	0.1	0.3	0.4	0.7
$(e_1\ e_2)$	0.1	0.3	0.4	0.7	$(e_1\ e_2)$	0.1	0.3	0.4	0.7
$(e_1\ e_3)$	0.1	0.3	0.4	0.6	$(e_1\ e_3)$	0.1	0.3	0.4	0.2
$(e_2\ e_1)$	0.2	0.4	0.6	0.4	$(e_2\ e_1)$	0.2	0.4	0.6	0.4
$(e_2\ e_2)$	0.1	0.4	0.4	0.4	$(e_2\ e_2)$	0.2	0.4	0.6	0.4
$(e_2\ e_3)$	0.1	0.4	0.4	0.4	$(e_2\ e_3)$	0.2	0.4	0.4	0.2
$(e_3\ e_1)$	0.3	0.5	0.6	0.1	$(e_3\ e_1)$	0.2	0.4	0.6	0.1
$(e_3\ e_2)$	0.1	0.7	0.4	0.1	$(e_3\ e_2)$	0.2	0.5	0.6	0.1
$(e_3\ e_3)$	0.1	0.6	0.4	0.1	$(e_3\ e_3)$	0.8	0.7	0.4	0.1
$(k_3\ k_2)$	x_1	x_2	x_3	x_4	$(k_3\ k_3)$	x_1	x_2	x_3	x_4
$(e_1\ e_1)$	0.2	0.5	0.5	0.8	$(e_1\ e_1)$	0.2	0.4	0.5	0.7
$(e_1\ e_2)$	0.1	0.6	0.4	0.7	$(e_1\ e_2)$	0.5	0.5	0.5	0.8
$(e_1\ e_3)$	0.1	0.6	0.4	0.6	$(e_1\ e_3)$	0.2	0.6	0.4	0.2
$(e_2\ e_1)$	0.1	0.5	0.3	0.6	$(e_2\ e_1)$	0.1	0.4	0.3	0.6
$(e_2\ e_2)$	0.1	0.7	0.3	0.6	$(e_2\ e_2)$	0.1	0.5	0.3	0.6
$(e_2\ e_3)$	0.1	0.6	0.3	0.6	$(e_2\ e_3)$	0.1	0.7	0.3	0.2
$(e_3\ e_1)$	0.3	0.5	0.4	0.5	$(e_3\ e_1)$	0.2	0.4	0.4	0.5
$(e_3\ e_2)$	0.1	0.6	0.4	0.5	$(e_3\ e_2)$	0.2	0.5	0.4	0.5
$(e_3\ e_3)$	0.1	0.6	0.4	0.5	$(e_3\ e_3)$	0.8	0.6	0.4	0.2

$H_C^* = F_A^* \wedge F_B^*$ obtained in Example 2 are shown below:

$$Max_a Min_b[H_C^*(a,b)] = \underset{a\in A}{\vee}\{\underset{b\in B}{\wedge}(F_A^*(a)\wedge F_B^*(b))\}$$

$$= \vee\begin{Bmatrix} \wedge\left\{F_{E_A}(k_1)\wedge F_{E_B}(k_2), F_{E_A}(k_1)\wedge F_{E_B}(k_3)\right\} \\ \wedge\left\{F_{E_A}(k_3)\wedge F_{E_B}(k_2), F_{E_A}(k_3)\wedge F_{E_B}(k_3)\right\} \end{Bmatrix}$$

$$= \vee\begin{Bmatrix} \wedge\left\{(k_1\ k_2),(k_1\ k_3)\right\} \\ \wedge\left\{(k_3\ k_2),(k_3\ k_3)\right\} \end{Bmatrix} = F_C(say) \tag{4}$$

where Table 3 show fuzzy soft set F_C.

$$Max_b Min_a[H_C^*(a,b)] = \underset{b\in B}{\vee}\{\underset{a\in A}{\wedge}(F_A^*(a)\wedge F_B^*(b))\}$$

$$= \vee\begin{Bmatrix} \wedge\left\{F_{E_A}(k_1)\wedge F_{E_B}(k_2), F_{E_A}(k_3)\wedge F_{E_B}(k_2)\right\} \\ \wedge\left\{F_{E_A}(k_1)\wedge F_{E_B}(k_3), F_{E_A}(k_3)\wedge F_{E_B}(k_3)\right\} \end{Bmatrix}$$

$$= \vee\begin{Bmatrix} \wedge\left\{(k_1\ k_2),(k_3\ k_2)\right\} \\ \wedge\left\{(k_1\ k_3),(k_3\ k_3)\right\} \end{Bmatrix} = F_D(say) \tag{5}$$

where Table 4 shows fuzzy soft set F_D.

APPLICATION IN DECISION MAKING PROBLEM

Suppose a set of projects are to be evaluated in two stages based on a certain set of parameters. In each stage two evaluators evaluate the projects and assign the marks between 0 and 100 and

Table 3. Fuzzy soft set F_C.

F_C	x_1	x_2	x_3	x_4
$(e_1\ e_1)$	0.2	0.4	0.5	0.7
$(e_1\ e_2)$	0.1	0.5	0.4	0.7
$(e_1\ e_3)$	0.1	0.6	0.4	0.2
$(e_2\ e_1)$	0.2	0.4	0.6	0.6
$(e_2\ e_2)$	0.1	0.5	0.4	0.6
$(e_2\ e_3)$	0.1	0.6	0.4	0.2
$(e_3\ e_1)$	0.2	0.4	0.6	0.5
$(e_3\ e_2)$	0.1	0.5	0.4	0.5
$(e_3\ e_3)$	0.1	0.6	0.4	0.2

Table 4. Fuzzy soft set F_D.

F_D	x_1	x_2	x_3	x_4
$(e_1\ e_1)$	0.1	0.3	0.4	0.7
$(e_1\ e_2)$	0.1	0.3	0.4	0.7
$(e_1\ e_3)$	0.1	0.3	0.4	0.6
$(e_2\ e_1)$	0.1	0.4	0.3	0.4
$(e_2\ e_2)$	0.4	0.4	0.3	0.4
$(e_2\ e_3)$	0.1	0.4	0.3	0.4
$(e_3\ e_1)$	0.3	0.5	0.4	0.1
$(e_3\ e_2)$	0.2	0.6	0.4	0.1
$(e_3\ e_3)$	0.8	0.6	0.4	0.1

problem here is to rank the projects based on the evaluation. Here, we present an algorithm to solve this decision making problem for which marks allotted are converted on the scale of 0 to 1 to get extended fuzzy soft sets and these sets will be the input for the proposed algorithm.

Algorithm:

Step 1: *Input extended fuzzy soft sets* F_A^* *and* F_B^*

Step 2: *Perform Cartesian AND product of* F_A^* *and* F_B^* *to obtain* H_C^*

Step 3: *Apply* $MaxMin$ *operators on* H_C^* *to obtain fuzzy soft sets* F_C *and* F_D

Step 4: *Apply* $MaxMin$ *operators on* F_C *and* F_D *to obtain four fuzzy sets*

Step 5: *Construct comparison matrix (Roy and Maji, 2007) of fuzzy sets obtained in step 4, in which both rows and columns are labeled by project names and the entries are (Kong et al., 2009)* $c_{ij} = \sum_{k=1}^{4}(\alpha_{ik} - \alpha_{jk})$ *where* α_{ik} *is the membership value of* i^{th}

Table 5. Marks allotted by evaluators k_1, k_2, k_3 and k_4.

k_1	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	80	60	40	60	80	70	60	20	50	80
e_2	70	50	60	70	70	70	70	20	50	70
e_3	60	90	50	80	40	40	40	30	40	70
k_2	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	60	80	70	70	70	40	60	20	10	90
e_2	10	50	70	60	50	30	80	40	10	70
e_3	80	60	60	50	80	70	10	30	10	80
k_3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	80	60	60	20	90	70	80	40	20	70
e_2	60	30	80	80	70	60	10	40	70	60
e_3	80	40	50	50	60	50	30	80	30	40
k_4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	50	40	70	50	80	80	80	60	70	20
e_2	60	70	50	30	80	70	80	40	90	60
e_3	70	60	50	90	60	50	80	30	50	70

project for k^{th} fuzzy set and then compute row sum $r_i = \sum_{j=1}^{4} c_{ij}$

Step 6: *The decision is $rank(x_i) > rank(x_j)$ if $r_i > r_j$ and $rank(x_i) = rank(x_j)$ if $r_i = r_j$*

Example 4

Suppose ten projects need to be evaluated based on a certain set of parameters by four evaluators in a pair in two different stages and projects to be ranked. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ be a list of projects and parameter (primary) set $E = \{e_1, e_2, e_3\}$. Let secondary parameter (evaluators) set $K = \{k_1, k_2, k_3, k_4\}$ and $A = \{k_1, k_2\}$ & $B = \{k_3, k_4\}$ be two pairs of evaluators. The marks allotted by the four evaluators are presented in Table 5.

Implementation of Algorithm

Step 1: Based on the evaluation of projects with respect to set parameters by the four evaluators, let the corresponding extended fuzzy soft sets be $F_A^* = \{(k_1, F_{E_A}(k_1)), (k_2, F_{E_A}(k_2))\}$ and $F_B^* = \{(k_3, F_{E_B}(k_3)), (k_4, F_{E_B}(k_4))\}$, where Table 6 shows the extended fuzzy soft sets F_A^* and F_B^*.

Step 2: Perform Cartesian *AND* Product of F_A^* and F_B^*:

$$H_C^* = F_A^* \wedge F_B^* = \begin{Bmatrix} \{(k_1 k_3)(F_{E_A}(k_1) \wedge F_{E_B}(k_3))\}, \{(k_1 k_4)(F_{E_A}(k_1) \wedge F_{E_B}(k_4))\}, \\ \{(k_2 k_3)(F_{E_A}(k_2) \wedge F_{E_B}(k_3))\}, \{(k_2 k_4)(F_{E_A}(k_2) \wedge F_{E_B}(k_4))\} \end{Bmatrix} \quad (6)$$

Step 3: *$MaxMin$ operators on "AND" products that is,* F_C = $Max_a Min_b[H_C^*(a,b)]$ *and* F_D = $Max_b Min_a[H_C^*(a,b)]$ *(Tables 7 and 8).*

Step 4: *Apply $MaxMin$ operators on F_C and F_D to get various fuzzy sets (Table 9).*

Step 5 and 6: *The Comparison table of the above fuzzy sets, row sum and ranking (Table 10).*

Table 6. Extended fuzzy soft sets F_A^* and F_B^*.

$F_{E_A}(k_1)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	0.8	0.6	0.4	0.6	0.8	0.7	0.6	0.2	0.5	0.8
e_2	0.7	0.5	0.6	0.7	0.7	0.7	0.7	0.2	0.5	0.7
e_3	0.6	0.9	0.5	0.8	0.4	0.4	0.4	0.3	0.4	0.7
$F_{E_A}(k_2)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	0.6	0.8	0.7	0.7	0.7	0.4	0.6	0.2	0.1	0.9
e_2	0.1	0.5	0.7	0.6	0.5	0.3	0.8	0.4	0.1	0.7
e_3	0.8	0.6	0.6	0.5	0.8	0.7	0.1	0.3	0.1	0.8
$F_{E_B}(k_3)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	0.8	0.6	0.6	0.2	0.9	0.7	0.8	0.4	0.2	0.7
e_2	0.6	0.3	0.8	0.8	0.7	0.6	0.1	0.4	0.7	0.6
e_3	0.8	0.4	0.5	0.5	0.6	0.5	0.3	0.8	0.3	0.4
$F_{E_B}(k_4)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
e_1	0.5	0.4	0.7	0.5	0.8	0.8	0.8	0.6	0.7	0.2
e_2	0.6	0.7	0.5	0.3	0.8	0.7	0.8	0.4	0.9	0.6
e_3	0.7	0.6	0.5	0.9	0.6	0.5	0.8	0.3	0.5	0.7
$(k_1\ k_3)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.8	0.6	0.4	0.2	0.8	0.7	0.6	0.2	0.2	0.7
$(e_1\ e_2)$	0.6	0.3	0.4	0.6	0.7	0.6	0.1	0.2	0.5	0.6
$(e_1\ e_3)$	0.8	0.4	0.4	0.5	0.6	0.5	0.3	0.2	0.3	0.4
$(e_2\ e_1)$	0.7	0.5	0.6	0.2	0.7	0.7	0.7	0.2	0.2	0.7
$(e_2\ e_2)$	0.6	0.3	0.6	0.7	0.7	0.6	0.1	0.2	0.5	0.6
$(e_2\ e_3)$	0.7	0.4	0.5	0.5	0.6	0.5	0.3	0.2	0.3	0.4
$(e_3\ e_1)$	0.6	0.6	0.5	0.2	0.4	0.4	0.4	0.3	0.2	0.7
$(e_3\ e_2)$	0.6	0.3	0.5	0.8	0.4	0.4	0.1	0.3	0.4	0.6
$(e_3\ e_3)$	0.6	0.4	0.5	0.5	0.4	0.4	0.3	0.3	0.3	0.4
$(k_1\ k_4)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.5	0.4	0.4	0.5	0.8	0.7	0.6	0.2	0.5	0.2
$(e_1\ e_2)$	0.6	0.6	0.4	0.3	0.8	0.7	0.6	0.2	0.5	0.6
$(e_1\ e_3)$	0.7	0.6	0.4	0.6	0.6	0.5	0.6	0.2	0.5	0.7
$(e_2\ e_1)$	0.5	0.4	0.6	0.5	0.7	0.7	0.7	0.2	0.5	0.2

Table 6. Contd.

$(e_2\ e_2)$	0.6	0.5	0.5	0.3	0.7	0.7	0.7	0.2	0.5	0.6
$(e_2\ e_3)$	0.7	0.5	0.5	0.7	0.6	0.5	0.7	0.2	0.5	0.7
$(e_3\ e_1)$	0.5	0.4	0.5	0.5	0.4	0.4	0.4	0.3	0.4	0.2
$(e_3\ e_2)$	0.6	0.7	0.5	0.3	0.4	0.4	0.4	0.3	0.4	0.6
$(e_3\ e_3)$	0.6	0.6	0.5	0.8	0.4	0.4	0.4	0.3	0.4	0.7
$(k_2\ k_3)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.6	0.6	0.6	0.2	0.7	0.4	0.6	0.2	0.1	0.7
$(e_1\ e_2)$	0.6	0.3	0.7	0.7	0.7	0.4	0.1	0.2	0.1	0.6
$(e_1\ e_3)$	0.6	0.4	0.5	0.5	0.6	0.4	0.3	0.2	0.1	0.4
$(e_2\ e_1)$	0.1	0.5	0.6	0.2	0.5	0.3	0.8	0.4	0.1	0.7
$(e_2\ e_2)$	0.1	0.3	0.7	0.6	0.5	0.3	0.1	0.4	0.1	0.6
$(e_2\ e_3)$	0.1	0.4	0.5	0.5	0.5	0.3	0.3	0.4	0.1	0.4
$(e_3\ e_1)$	0.8	0.6	0.6	0.2	0.8	0.7	0.1	0.3	0.1	0.7
$(e_3\ e_2)$	0.6	0.3	0.6	0.5	0.7	0.6	0.1	0.3	0.1	0.6
$(e_3\ e_3)$	0.8	0.4	0.5	0.5	0.6	0.5	0.1	0.3	0.1	0.4
$(k_2\ k_4)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.5	0.4	0.7	0.5	0.7	0.4	0.6	0.2	0.1	0.2
$(e_1\ e_2)$	0.6	0.7	0.5	0.3	0.7	0.4	0.6	0.2	0.1	0.6
$(e_1\ e_3)$	0.6	0.6	0.5	0.7	0.6	0.4	0.6	0.2	0.1	0.7
$(e_2\ e_1)$	0.1	0.4	0.7	0.5	0.5	0.3	0.8	0.4	0.1	0.2
$(e_2\ e_2)$	0.1	0.5	0.5	0.3	0.5	0.3	0.8	0.4	0.1	0.6
$(e_2\ e_3)$	0.1	0.5	0.5	0.6	0.5	0.3	0.8	0.3	0.1	0.7
$(e_3\ e_1)$	0.5	0.4	0.6	0.5	0.8	0.7	0.1	0.3	0.1	0.2
$(e_3\ e_2)$	0.6	0.6	0.5	0.3	0.8	0.7	0.1	0.3	0.1	0.6
$(e_3\ e_3)$	0.7	0.6	0.5	0.5	0.6	0.5	0.1	0.3	0.1	0.7

RESULTS AND DISCUSSION

The performance of the algorithm is illustrated with an example of ranking ten different projects which are evaluated by four evaluators based on three different parameters. The ranking strategy is based on value of row sum (r_i) as depicted in Table 10. The project with highest row sum (r_i) is ranked number 1 and the project with lowest row sum (r_i) is given last rank. For the example under discussion, project x_5 has the highest row sum that is, 6.2 and is given rank 1 and the project x_9 has the least row sum and hence assigned last rank. If two or more values in the row sum are same then the corresponding projects will be assigned the same ranks

Table 7. Fuzzy soft set $F_C = Max_a Min_b [H_C^*(a,b)]$.

F_C	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.5	0.4	0.6	0.2	0.8	0.7	0.6	0.2	0.2	0.2
$(e_1\ e_2)$	0.6	0.3	0.5	0.3	0.7	0.6	0.1	0.2	0.5	0.6
$(e_1\ e_3)$	0.7	0.4	0.5	0.5	0.6	0.5	0.3	0.2	0.3	0.4
$(e_2\ e_1)$	0.5	0.4	0.6	0.2	0.7	0.7	0.8	0.4	0.2	0.2
$(e_2\ e_2)$	0.6	0.3	0.5	0.3	0.7	0.6	0.1	0.4	0.5	0.6
$(e_2\ e_3)$	0.7	0.4	0.5	0.5	0.6	0.5	0.3	0.3	0.3	0.4
$(e_3\ e_1)$	0.5	0.4	0.6	0.2	0.8	0.7	0.4	0.3	0.2	0.2
$(e_3\ e_2)$	0.6	0.3	0.5	0.3	0.7	0.6	0.1	0.3	0.4	0.6
$(e_3\ e_3)$	0.7	0.4	0.5	0.5	0.6	0.5	0.3	0.3	0.3	0.4

Table 8. Fuzzy soft set $F_D = Max_b Min_a [H_C^*(a,b)]$.

F_D	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$(e_1\ e_1)$	0.6	0.6	0.4	0.5	0.7	0.4	0.6	0.2	0.1	0.7
$(e_1\ e_2)$	0.6	0.6	0.4	0.6	0.7	0.4	0.6	0.2	0.1	0.6
$(e_1\ e_3)$	0.6	0.6	0.4	0.6	0.6	0.4	0.6	0.2	0.1	0.7
$(e_2\ e_1)$	0.1	0.5	0.6	0.5	0.5	0.3	0.7	0.2	0.1	0.7
$(e_2\ e_2)$	0.1	0.5	0.6	0.6	0.5	0.3	0.7	0.2	0.1	0.6
$(e_2\ e_3)$	0.1	0.5	0.5	0.6	0.5	0.3	0.7	0.2	0.1	0.7
$(e_3\ e_1)$	0.6	0.6	0.5	0.5	0.4	0.4	0.1	0.3	0.1	0.7
$(e_3\ e_2)$	0.6	0.6	0.5	0.5	0.4	0.4	0.1	0.3	0.1	0.6
$(e_3\ e_3)$	0.6	0.6	0.5	0.5	0.4	0.4	0.1	0.3	0.1	0.7

Table 9. Various fuzzy sets after applying $MaxMin$ operators on F_C and F_D.

Fuzzy sets	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$Max_a Min_b (F_C)$	0.5	0.3	0.5	0.2	0.6	0.5	0.1	0.3	0.2	0.2
$Max_b Min_a (F_C)$	0.7	0.4	0.6	0.5	0.7	0.7	0.4	0.2	0.4	0.6
$Max_a Min_b (F_D)$	0.6	0.6	0.5	0.5	0.6	0.4	0.7	0.3	0.1	0.6
$Max_b Min_a (F_D)$	0.1	0.5	0.4	0.5	0.4	0.3	0.1	0.2	0.1	0.7

Table 10. Comparison table, row sum and ranking.

Row sum	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	Row sum (r_i)	Rank
x_1	0.0	0.1	-0.1	0.2	-0.4	0.0	0.6	0.9	1.1	-0.2	**2.2**	4
x_2	-0.1	0.0	-0.2	0.1	-0.5	-0.1	0.5	0.8	1.0	-0.3	1.2	5
x_3	0.1	0.2	0.0	0.3	-0.3	0.1	0.7	1.0	1.2	-0.1	**3.2**	3
x_4	-0.2	-0.1	-0.3	0.0	-0.6	-0.2	0.4	0.7	0.9	-0.4	0.2	6
x_5	0.4	0.5	0.3	0.6	0.0	0.4	1.0	1.3	1.5	0.2	**6.2**	1
x_6	0.0	0.1	-0.1	0.2	-0.4	0.0	0.6	0.9	1.1	-0.2	**2.2**	4
x_7	-0.6	-0.5	-0.7	-0.4	-1.0	-0.6	0.0	0.3	0.5	-0.8	-3.8	7
x_8	-0.9	-0.8	-1.0	-0.7	-1.3	-0.9	-0.3	0.0	0.2	-1.1	-6.8	8
x_9	-1.1	-0.6	-1.2	-0.9	-1.5	-1.1	-0.5	-0.2	0.0	-1.3	-8.4	9
x_{10}	0.2	0.3	0.1	0.4	-0.2	0.2	0.8	1.1	1.3	0.0	**4.2**	2

which can be seen for projects x_1 and x_6.

CONCLUSION

Here, a new approach for decision making problem is introduced by extending the definition of fuzzy soft set for multiple parameter sets called extended fuzzy soft set. Some operations such as “AND” and “$MaxMin$” are defined. Finally an algorithm for decision making was presented and the decision making process includes construction of comparison matrix and ranking strategy based on the row sum of comparison matrix. The proposed algorithm is illustrated with an example where ten different projects are evaluated and ranked based on the marks allotted by four different evaluators.

Data availability

The data used in this study is a randomly generated data to validate the algorithm presented in the paper and is not real time data.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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Continuous frame in Hilbert space and its applications

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In this paper, we study continuous frames in Hilbert spaces using a family of linearly independent vectors called coherent state (CS) and applying it in any physical space. To accomplish this goal, the standard theory of frames in Hilbert spaces, using discrete bases, is generalized to one where the basis vectors may be labeled using discrete, continuous or a mixture of the two types of indices. A comprehensive analysis of such frames is presented and illustrated by the examples drawn from a toy example Sea Star and the affine group.

Key words: Frame, continuous frame, unitary representation, coherent state (CS), sea star, affine group.

INTRODUCTION

The Hilbert space is the natural framework for the mathematical description of many areas of Physics and Mathematics. The most economical solution, is of course to use an orthonormal basis, $\{\phi_n, n \in \mathrm{N}\}$ which gives in addition the uniqueness of decomposition:

$$\phi = \sum_{n \in N} \langle \phi_n | \phi \rangle \phi_n . \quad (1)$$

The uniqueness of the decomposition and the orthogonality of the basis vectors, while maintaining its others useful properties: fast convergence, numerical stability of the reconstruction $\{\langle \phi_n | \phi \rangle\} \rightarrow \phi$, etc. The resulting object is called a discrete frame, a concept introduced by Duffin and Schaeffer (1952) in the context of non-harmonic Fourier analysis. Latter the concept of generalization of frames was proposed by Daubechies et al. (1986) and then independently by Ali et al., (1993). Let H be an abstract, separable Hilbert space (over the complexes C) and $GL(H)$ the group of all bounded linear operators on H which have bounded inverses, throughout this paper.

The definition is very simple: a family of vectors $\{\phi_n, n \in N\} \subset H$ (the Hilbert space) is a frame if there are two constants $A, B > 0$ such that, for all $\phi \in \mathrm{H}$, one has:

$$A \| \phi \|^2 \leq \sum_{n \in N} |\langle \phi_n | \phi \rangle|^2 \leq B \| \phi \|^2 \quad (2)$$

where the constant A and B are called frame bounds and if $A = B$, then the frame is called tight. Writing

$$S\phi = \sum_{n \in N} \langle \phi_n | \phi \rangle \phi_n , \quad (3)$$

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Here, S is a positive bounded operator with bounded inverse S^{-1}. Additionally, the set $\{\phi_n^{'}, n \in N\}$, where $\phi_n^{'} = S^{-1}\phi_n$, is called dual or reciprocal frame, with frame bounds B^{-1}, A^{-1}. Combining the two allows for the recovery of any element ϕ from its frame components:

$$\phi = \sum_{n\in N}\langle \phi_n^{'} | \phi\rangle \phi_n = \sum_{n\in N}\langle \phi_n | \phi\rangle \phi_n^{'} \quad (4)$$

In mathematical physics, the coherent states, $\eta_g = U(g)\eta$ actually yield the following continuous resolution of the identity on the group, G with the (left) Haar measure dg :

$$\int_G |\eta_g\rangle \langle \eta_g| dg = I, \quad (5)$$

where U being a strongly continuous unitary representation of G on the H, and η a fixed, suitable vector in H. Rewriting Equation 5, we obtain:

$$\int_G |\langle \eta_g | \phi\rangle|^2 = \|\phi\|^2, \ \forall \phi \in H \quad (6)$$

Analogy with a tight frame is clear and it seems natural to call the set of vectors $\{\eta_g, g \in G\}$ a continuous tight frame.

One should mention furthermore that the theory of frames has been elaborated by Ali et al. (1993, 2000) and Daubechies et al. (1986) and Daubechies (1990). Moreover, the interested reader can refer to Antoine and Grossmann (1976), Duffin and Schaeffer (1952), Gazeau (2016), and Christensen (2003). In this paper, continuous frames in Hilbert space were studied and applied to any physical space.

MATERIALS AND METHODS

Here, mathematical formulation of frames in H was discussed by taking the famous articles by Ali et al., (1993) and Rahimi et al., (2017).

Mathematical formulation of frames

A frame H is the union of a choice of linearly independent sets of vectors, satisfying a specific completeness-or rather over completeness-condition. Each set of vectors could be labeled by a set of discrete, continuous, or mixed indices. Let X be a locally compact space (which could be partly, or completely discrete) and let ν be a regular Borel measure on X with support equal to X. A set of vectors $\eta_x^i \in H, \quad i = 1,2,\ldots\ldots\ldots, n < \infty, \ x \in X$ is said to constitute a rank n frame F if the following conditions hold:

(i) For all $x \in X$, $\{\eta_x^i, i = 1,2,\ldots\ldots n\}$ is a linearly independent set; and

(ii) There exists a positive operator $A \in GL(H)$ such that,

$$\sum_{i=1}^{n}\int_X |\eta_x^i\rangle \langle \eta_x^i| d\nu(x) = A, \quad (7)$$

the integral converging weakly.

To be more explicit, we shall denote the frame so defined by $F\{\eta_x^i, A, n\}$. Note that if $X = J$ is a discrete set and ν is a counting measure, then the condition of Equation 7 yields the following equation.

$$\sum_{i=1}^{n}\sum_{j\in J}|\eta_j^i\rangle\langle \eta_j^i| = A \quad (8)$$

Or simply,

$$\sum_{k\in J}|\eta_k\rangle\langle \eta_k| = A \quad (9)$$

(J ' being another discrete index set). It is in this form that the definition of a frame is conventionally given (Duffin and Schaeffer, 1952; Fornasier and Rauhut, 2005) such that Equation is an obvious generalization.

Let $\sigma(A)$ be the spectrum of the self-adjoint (positive) operator A and let $m(A)$ and $M(A)$ be its infimum and supremum, respectively,

$$m(A) = \inf_{\|\phi\|=1}\langle \phi | A\phi\rangle \neq 0$$
$$M(A) = \sup_{\|\phi\|=1}\langle \phi | A\phi\rangle \neq 0 \quad (10)$$

$\phi \in H$, so that $m(A), M(A) \in \sigma(A)$ and

$$\sigma(\mathrm{A}) \subset [m(\mathrm{A}), \mathrm{M}(\mathrm{A})] \quad (11)$$

It is then clear that $\forall \phi \in H$ (Equation 7), implies the usual frame condition:

$$m(A)\|\phi\|^2 \leq \sum_{i=1}^{n}\int_X |\langle \eta_x^i | \phi\rangle|^2 d\nu(x) \leq M(A)\|\phi\|^2 \quad (12)$$

In other words, $m(A)$ and $M(A)$ are the frame bounds of common parlance. Furthermore, $M(A)^{-1}$ and $m(A)^{-1}$ are the infimum and supremum of $\sigma(A^{-1})$ and both $M(A)^{-1}, m(A)^{-1} \in \sigma(A^{-1})$, with

$$\sigma\left(A^{-1}\right) \subset \left[M(A)^{-1}, m(A)^{-1}\right] \qquad (13)$$

Defining

$$\eta_x'^{i} = A^{-1}\eta_x^{i} \qquad (14)$$

$i = 1, 2, \cdots, n, \; x \in X,$ we easily verify that

$$\sum_{i=1}^{n} \int_X \left| \eta_x'^{i} \right\rangle \left\langle \eta_x'^{i} \right| d\nu(x) = A^{-1}, \qquad (15)$$

Satisfying the frame condition

$$M(A)^{-1}\|\phi\|^2 \le \sum_{i=1}^{n} \int_X \left| \left\langle \eta_x'^{i} \middle| \phi \right\rangle \right|^2 d\nu(x) \le m(A)^{-1}\|\phi\|^2 \qquad (16)$$

$\forall \phi \in H$. The frame $F\left\{\eta_x'^{i}, A^{-1}, n\right\}$ is said to be the dual frame of $F\left\{\eta_x^{\,i}, A, n\right\}$. The width or snugness of the frame $F\left\{\eta_x^{\,i}, A, n\right\}$ is as follows:

$$w(F) = \frac{M(A) - m(A)}{M(A) + m(A)} \qquad (17)$$

Obviously, $0 \le w(F) < 1$ and $w(F)$ measure the spectral width of the operator A. If $w(F) = 0$, that is, if $A = \lambda I$ ($\lambda > 0$ and $I =$ the identity operator on H), then the frame F is called tight. Note that a frame $F\left\{\eta_x^{\,i}, A, n\right\}$ and its dual $F\left\{\eta_x^{\,i'}, A^{-1}, n\right\}$ have the same width and the frame is self-dual iff $A = I$ associated naturally to the frame $F\left\{\eta_x^{\,i}, A, n\right\}$, there is a self-dual, tight frame $F\left\{\overline{\eta}_x^{\,i}, I, n\right\}$, with:

$$\overline{\eta}_x^{\,i} = A^{-1/2}\eta_x^{\,i}, \qquad (18)$$

In fact, if T is any operator $GL(H)$ and T^* its adjoint, then writing

$$\overline{\eta}_x^{\,i} = T\eta_x^{\,i}, \qquad \overline{A} = TAT^*, \qquad (19)$$

We see that

$$\sum_{i=1}^{n} \int_X \left| \overline{\eta}_x^{i} \right\rangle \left\langle \overline{\eta}_x^{i} \right| d\nu(x) = \overline{A} \qquad (20)$$

Such that we obtain $F\left\{\overline{\eta}_x^{\,i}, \overline{A}, n\right\}$. In particular, if T is a unitary operator $\left(TT^* = T^*T = I\right)$, then $F\left\{\eta_x^{\,i}, A, n\right\}$ and $F\left\{\overline{\eta}_x^{\,i}, \overline{A}, n\right\}$ are said to be unitarily equivalent frames. In this case, of course, $\sigma(A) = \sigma(\overline{A})$ and the frame widths, $w\left(F\left\{\eta_x^{\,i}, A, n\right\}\right) = w\left(F\left\{\overline{\eta}_x^{\,i}, \overline{A}, n\right\}\right)$. Note, however, that Equation 18 is not the only way to obtain a self-dual tight frame from $F\left\{\eta_x^{\,i}, A, n\right\}$. Indeed, if U is any unitary operator on H, then since we can always write $A = A^{1/2}U^*UA^{1/2}$, we see that with:

$$\overline{\eta}_x^{\,i} = UA^{-1/2}\eta_x^{\,i} \qquad (18a)$$

Another self-dual tight frame $F\left\{\overline{\eta}_x^{\,i}, I, n\right\}$ is obtained which is unitarily equivalent to $F\left\{\eta_x^{i}, I, n\right\}$.

There is a sense in which any two frames $F\left\{\eta_x^{\,i}, A, n\right\}$ and $F\left\{\overline{\eta}_x^{\,i}, \overline{A}, n\right\}$, related by Equation 19, are equivalent and we proceed to study this point a little more closely. If we introduce the positive operator:

$$F(x) = \sum_{i=1}^{n} \left| \eta_x^{\,i} \right\rangle \left\langle \eta_x^{\,i} \right|, \qquad (21)$$

For each $x \in X$, Equation 7 assumes the form:

$$\int_X F(x)\, d\nu(x) = A \qquad (22)$$

Of course, for each x there is more than one choice of linearly independent vectors η_x^{i} for which Equation 21 is satisfied. The arbitrariness is quantified by Ali et al., (1993) and Friedberg et al., (2003).

Theorem 1. *If* $\left\{\overline{\eta}_x^{i}, i = 1, 2, \ldots\ldots, n\right\}$ is linearly independent set of vectors for which

$$F(x) = \sum_{i=1}^{n} \left| \overline{\eta}_x^{\,i} \right\rangle \left\langle \overline{\eta}_x^{\,i} \right|, \qquad (23)$$

if there exists an $n \times n$ unitary matrix $U(x)$, with elements $U_{ij}(x)$, such that

$$\overline{\eta}_x^{\,i} = \sum_{j=1}^{n} U_{ij}(x)\eta_x^{j}, \quad i = 1, 2, \ldots\ldots\ldots, n. \qquad (24)$$

Proof

It is clear from the unitary of $U(x)$ (that is, from $\sum_{j=1}^{n}\overline{U_{ij}(x)}U_{kj}(x)=\delta_{ik}$) that if $\overline{\eta}_x^i$ and η_x^i are related by Equation 24, then Equation 23 holds. On the other hand, assume that $\overline{\eta}_x^i$ are linearly independent and satisfy Equation 23. Then, for all $\phi\in H$,

$$\langle\phi|F(x)\phi\rangle=\sum_{i=1}^{n}\left|\langle\eta_x^i|\phi\rangle\right|^2=\sum_{i=1}^{n}\left|\langle\overline{\eta}_x^i|\phi\rangle\right|^2. \quad (25)$$

Setting

$$z_i=\langle\eta_x^i|\phi\rangle\in C$$
$$z_i'=\langle\overline{\eta}_x^i|\phi\rangle\in C \quad (26)$$

This equation becomes

$$\sum_{i=1}^{n}|z_i|^2=\sum_{i=1}^{n}|z_i'|^2. \quad (27)$$

Now let $P(x)$ be the projection operator onto the range of $F(x)$ (and hence $P(x)$ is also its support). Then both $\{\eta_x^i\}_{i=1}^n$ and $\{\overline{\eta}_x^i\}_{i=1}^n$ span the subspace $P(x)H$ of H, and there exists an $n\times n$ invertible matrix $U(x)$ for which

$$\overline{\eta}_x^i=\sum_{j=1}^{n}U_{ij}(x)\eta_x^j.$$

Thus, for all $\phi\in H$,

$$\langle\overline{\eta}_x^i|\phi\rangle=\sum_{j=1}^{n}U_{ij}(x)\langle\eta_x^j|\phi\rangle$$
$$\Rightarrow z_i'=\sum_{j=1}^{n}U_{ij}(x)z_j,$$

and, hence by Equation 27, $U(x)$ is unitary.

RESULTS AND DISCUSSION

Applications of continuous frames

Here, we discuss two useful examples on the continuous frame in Hilbert space (Ali et al., 2000, 1993; Gazeau (2016); Rahimi et al., (2017); Friedman A (1970).

Example 1: A toy example - Sea star

Consider the Euclidean plane with Dirac notations

$$\vec{j}\equiv\left|\tfrac{\pi}{2}\right\rangle\equiv\begin{pmatrix}0\\1\end{pmatrix}$$
$$\langle 0|0\rangle=1=\left\langle\tfrac{\pi}{2}\middle|\tfrac{\pi}{2}\right\rangle,\left\langle 0\middle|\tfrac{\pi}{2}\right\rangle=0$$
$$I=|0\rangle\langle 0|+\left|\tfrac{\pi}{2}\right\rangle\left\langle\tfrac{\pi}{2}\right|$$
$$\Leftrightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

In the Euclidean plane, $\Re^2$ orthonormal basis is defined by the two ket vectors $|0\rangle$ and $\left|\tfrac{\pi}{2}\right\rangle$, where $|\theta\rangle$ denotes the unit vector with polar angle $\theta\in[0,2\pi)$. This frame is such that

$$\langle 0|0\rangle=1=\left\langle\tfrac{\pi}{2}\middle|\tfrac{\pi}{2}\right\rangle,\left\langle 0\middle|\tfrac{\pi}{2}\right\rangle=0$$

and the resolution of the identity comes through the sum of their corresponding orthogonal projections,

$$I=|0\rangle\langle 0|+\left|\tfrac{\pi}{2}\right\rangle\left\langle\tfrac{\pi}{2}\right|$$
$$\Leftrightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

Now, coherent state $=\left|\frac{2n\pi}{5}\right\rangle\equiv R\left(\frac{2n\pi}{5}\right)|0\rangle$, where $n=0,1,2,3,4\,\mathrm{mod}(5)$ and matrix representation is

$$R(\theta)=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}.$$

To the unit vector $|\theta\rangle=\cos\theta|0\rangle+\sin\theta\left|\tfrac{\pi}{2}\right\rangle$ corresponds with the orthogonal projector p_θ given by

$$p_\theta=|\theta\rangle\langle\theta|=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\end{pmatrix}$$
$$=\begin{pmatrix}\cos^2\theta&\cos\theta\sin\theta\\\cos\theta\sin\theta&\sin^2\theta\end{pmatrix}$$
$$=R(\theta)|0\rangle\langle 0|R(-\theta)$$

Again the resolution of the identity for Sea Star is as follows:

$$\frac{2}{5}\sum_{n=0}^{4}\left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

Here, $X = \{0,1,2,3,4\}$ ≡is the set of orientations ≡ angles $\frac{2\pi n}{5}$ explored by the starfish which is equipped with uniform weight $2/5$. The operator

$p_n = \left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right|$ acts on $H = \Re^2$.

From $p_n = \left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right|$ acts on $H = \Re^2$ and given a $n_0 \in \{0,1,2,3,4\}$ one derives the probability distribution on $X = \{0,1,2,3,4\}$

$$tr(p_{n_0} p_n) = \left|\left\langle\frac{2\pi n_0}{5}\middle|\frac{2\pi n}{5}\right\rangle\right|^2 = \cos^2\left(\frac{2\pi(n_0 - n)}{5}\right)$$

Choosing $n_0 = 0$, values are

$$tr(p_0 p_n) = \left(\frac{1}{2\tau}\right)^2 \approx 0.0955 \text{ for } n = 1,4$$

$$tr(p_0 p_n) = \left(-\frac{\tau}{2}\right)^2 \approx 0.6545 \text{ for } n = 2,3$$

with $\tau = (1+\sqrt{5})/2 \approx 1.618$ (golden mean). Check from $\tau^2 = \tau + 1$ that

$$\frac{2}{5}\left(1 + \frac{1}{4\tau^2} + \frac{\tau^2}{4}\right) = 1$$

Projectors $n \mapsto p_n$ with Hilbert-Schmidt norm $\|p_n\|^2 = tr(p_n p_n^*) = tr(p_n^2) = 1$ allow a localization distance on X to be defined:

$$d_{HS}(n,n') = \|p_n - p_{n'}\| = \sqrt{tr(p_n - p_{n'})^2} = \sqrt{2}\left|\sin\frac{2\pi(n-n')}{5}\right|$$

Similarly, any regular N-fold polygon in the plane have satisfied the resolution of unity by the following way:

$$\frac{2}{N}\sum_{n=0}^{N}\left|\frac{2\pi n}{N}\right\rangle\left\langle\frac{2\pi n}{N}\right| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Finally, if we consider the continuous case, then we have,

$$\frac{1}{\pi}\int_0^{2\pi} d\theta |\theta\rangle\langle\theta| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Example 2: The affine group

In order to obtain a concrete situation where the more general considerations of the results do indeed apply, let us construct a rather unorthodox family of CS for the affine group G_A. The connected affine group consists of transformation of $\Re$ of the following type:

$$x \mapsto ax + b,\ x \in \Re \tag{28}$$

where, $a > 0, b \in R$. Writing

$$g = (a,b) \in G_A, \tag{29}$$

We have the multiplication law,

$$g_1 g_2 = (a_1 a_2, b_1 + a_1 b_2). \tag{30}$$

The matrix representation

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}; a \neq 0, b \in R$$

reproduces this composition rule. The inverse is given by the matrix

$$(a,b)^{-1} = \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ 0 & 1\end{pmatrix} = \left(\frac{1}{a}, -\frac{b}{a}\right)$$

If we take a vector $\begin{pmatrix}x\\ 1\end{pmatrix}$, then trivially the action of the matrix Equation 30 on this vector is given by

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix} = \begin{pmatrix}ax+b\\ 1\end{pmatrix}$$

which exactly reproduces the action $x \to ax+b$ on $\Re$. It is clear that this class of matrices constitute a group, called the full affine group which is denoted by G_A but note that, it is not a connected group. So that, if we consider only those matrices from the class of matrices where $a>0$, then the class of matrices forms a subgroup of G_A which is denoted by $G_A{}^+$.

On the Hilbert space

$$H_n = L^2(R^+, \mu_n),\ n = \text{integer} \geq 1$$

$$d\mu_n(r) = r^{n+1}dr, \tag{31}$$

G_A has the unitary irreducible representation $g \mapsto U_n(g)$ given by

$$(U_n(a,b)\psi)(r) = a^{n/2}e^{ibr}\psi(ar),\ \psi \in H_n \tag{32}$$

Consider the subgroup H of G,

$$H = \{g \in G_A \mid g = (a,0), a \in \Re^+\}. \tag{33}$$

Then $G_A/H \approx \Re^+,\ \forall(a,b) \in G_A,$

$$(a,b) = (1,b)(a,0), \qquad b \in \Re \tag{34}$$

Also, since $\forall x \in \Re,$

$$(a,b)(1,x) = (1, ax+b)(a,0), \tag{35}$$

On the coset space G_A/H, the action of G_A in $\Re$, can be written as:

$$gx = ax+b, \qquad g = (a,b) \in G_A \tag{36}$$

On G_A/H, so parameterized, we choose the quasi-invariant measure,

$$dv = dx, \tag{37}$$

so that

$$\frac{dv_g}{dv}(x) \equiv \lambda(g,x) = a, g = (a,b) \tag{38}$$

Choosing a (global) section, $\sigma : \Re \to G_A$

$$\sigma(x) = (1,x), \tag{39}$$

we get

$$\lambda(\sigma(x), x) = 1, \tag{40}$$

so that coherent states may be constructed[14] for a suitable choice of $\eta \in H_n$ as

$$\eta_x = \lambda(\sigma(x), x)U_n(\sigma(x))\eta, x \in \Re \tag{41}$$

Thus,

$$\eta_x(r) = e^{ixr}\eta(r),\ r \in \Re^+ \tag{42}$$

The operator,

$$A = \int_R |\eta_x\rangle\langle\eta_x| dx, \tag{43}$$

is easily computed to be a multiplication operator on H_n,

$$(\mathrm{A}\phi)(r) = 2\pi r^n\ |\eta(r)|^2 \phi(x,)\ \forall\phi \in H_n \tag{44}$$

In order for A to be bounded and invertible with $\|A\| = 1,$ we must, therefore, impose on $\eta \in H_n$ the conditions,

$$(i) \quad \sup_{r \in R^+}\left[2\pi r^{n-1}|\eta(r)|^2\right] = 1 \tag{45}$$

$(ii)\ |\eta(r)|^2 \neq 0,$ exceptperhaps at isolated points

$$r \in \Re^+ \tag{46}$$

These conditions, together with the fact that $\eta \in H_n$ that is,

$$\int_{R^+} |\eta(r)|^2 d\mu_n < \infty, \tag{47}$$

imply that A^{-1} is unbounded. In fact, since

$$(\mathrm{A}^{-1}\phi)(r) = \frac{1}{2\pi}\frac{r^{1-n}}{|\eta(r)|^2}\phi(r), \tag{48}$$

ϕ lies in the set $D\left(A^{-1}\right)$ iff

$$\frac{1}{(2\pi)^2}\int_{R^+}\frac{r^{2-2n}}{|\eta(r)|^4}|\phi(r)|^2 d\mu_n(r)<\infty. \quad (49)$$

Thus, the representation U_n of G_A is square-integrable mod(H,σ) and coherent states (Equation 42) may be constructed for any $\eta\in H_n$ satisfying the admissibility conditions (Equations 45 and 46). However, the coherent states do not define a frame, since A^{-1} is unbounded. In fact from Equations 45 and 47, it follows that none of the vectors η_x is the domain of either $\mathrm{A}^{-1/2}$ or A^{-1}. The map

$$W_\eta : H_n \to L^2(\Re, dx),$$

$$\left(W_\eta\varphi\right)(x)=\langle\eta_x|\varphi\rangle=\int_{\Re^+} e^{irx}\overline{\eta(r)}\varphi(r)\,d\mu_n(r), \quad (50)$$

is clearly bounded and its range in $L^2(R,dx)$ is closed in the norm

$$\|\Phi\|_\eta^2=\left\langle\Phi\middle|A_\eta^{-1}\Phi\right\rangle_{L^2(\Re,dx)}, \quad (51)$$

where A_η^{-1} is the image, on $L^2(\Re,dx)$, of A^{-1} under W_η. The evaluation maps $E_\eta(x): H_\eta \to C$, given by $\Phi\mapsto\Phi(x)$, is easily seen to be continuous (in the Equations 51 norm). However, the reproducing kernel,

$$\mathrm{K}(x,y)=\left\langle\eta_x\middle|\mathrm{A}^{-1}\eta_y\right\rangle=\frac{1}{2\pi}\int_{R^+} e^{i(x,y)r}dr, \quad (52)$$

is a distribution which defines a sesquilinear form on H_η.

Conclusion

An introductory-level theory of continuous frames in Hilbert space was studied, focusing primarily on the analysis and ending with its applications to possible physical space. The mathematical construction of frames was illustrated by the examples drawn from a toy example Sea Star and the affine group.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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